APPENDIX C

Math Self-Test Solutions

Score 4 points for each correct answer. For an assessment of your total, read the part before the start of the test.

Use the following if you need to (some are approximations): $\sin 30° = \cos 60° = 0.5$, $\sin 45° = \cos 45° = 0.707$, $\sin 60° = \cos 30° = 0.866$, $\pi = 3.141$, $\sqrt{2} = 1.414$, $\sqrt{3} = 1.732$.

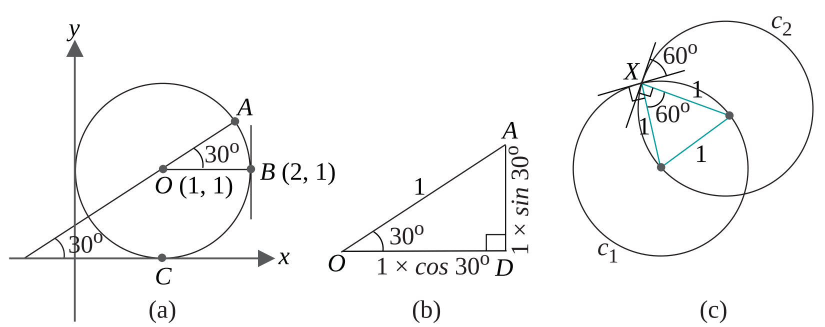

Figure C.1: (a) Circle of unit radius (b) Right-angled triangle (c) Two circles of unit radius.

The first seven questions refer to Figure C.1(a).

1. What is the equation of the circle?

   Answer: Generally, the equation of a circle on the xy-plane centered at (a, b) and of radius r is

$$(x - a)^2 + (y - b)^2 = r^2$$

So the equation of the drawn circle is

$$(x-1)^2+(y-1)^2=1^2 \quad \text{which evaluates to} \quad x^2+y^2-2x-2y+1=0$$

2. What are the coordinates of point C?

   Answer: $(1,0)$.

3. What is the equation of the tangent to the circle at B?

   Answer: $x=2$.

4. What is the length of the short arc of the circle from A to B?

   Answer: The angle subtended at the center by this arc is $\angle AOB = 30°$ (Figure C.1(a)).

   Therefore, the length of the arc is $\frac{30}{360}$ of the circumference $= \frac{30}{360} \times 2\pi \times 1 = \frac{\pi}{6} = 3.141/6 = 0.5235$.

5. What are the coordinates of point A?

   Answer: Suppose the horizontal line through O and the vertical line through A intersect at D (Figure C.1(b)). The hypotenuse OA of the right-angled triangle ODA is of length 1 (= radius of the circle). Therefore, the length of $OD = 1 \times \cos 30° = 0.866$ and the length of $AD = 1 \times \sin 30° = 0.5$.

   These lengths are the displacements of A in the x- and y-direction, respectively, from O. Therefore, the coordinates of A are $(1,1) + (0.866, 0.5) = (1.866, 1.5)$.

6. If the circle is moved (without turning) so that its center lies at $(-3,-4)$, where then does point B lie?

   Answer: Since the center is originally at $(1,1)$ the translation that moves it to $(-3,-4)$ consists of a displacement of -4 in the x-direction and -5 in the y direction. The same translation applies to B, so B moves $(2,1)+(-4,-5)=(-2,-4)$.

7. Suppose another circle is drawn with center at A and passing through O. The two circles intersect in two points. What angles do their tangents make at these points?

   Answer: The two circles c_1 and c_2 intersect at points X and Y (Figure C.1(c)). The triangle OAX is equilateral as all its sides are of length 1, the radius of either circle. All its angles, therefore, are 60°.

   Now, the tangents to c_1 and c_2 at X are perpendicular, respectively, to OX and AX. Therefore, the angle between them is the same as that between OX and AX, which is 60°.

Symmetrically, the tangents to the two circles at Y intersect at $60°$ as well.

8. If a straight line on a plane passes through the points $(3, 1)$ and $(5, 2)$, which, if any, of the following two points does it pass through as well: $(9, 4)$ and $(12, 6)$?

   Answer: Suppose the equation of the straight line is $y = mx + c$ (any non-vertical straight line has an equation of this form, called the slope-intercept form). Since it passes through $(3, 1)$ and $(5, 2)$, we have the two equations

$$\begin{aligned} 1 &= 3m + c \\ 2 &= 5m + c \end{aligned}$$

   Solving simultaneously, we get $m = \frac{1}{2}$ and $c = -\frac{1}{2}$, yielding the equation $y = \frac{1}{2}x - \frac{1}{2}$ for the line. Of the two points $(9, 4)$ and $(12, 6)$, only $(9, 4)$ satisfies this equation and so lies on it.

9. What are the coordinates of the midpoint of the straight line segment joining the points $(3, 5)$ and $(4, 7)$?

   Answer: The coordinates of the midpoint are $(\frac{3+4}{2}, \frac{5+7}{2}) = (3.5, 6)$.

10. At what point do the straight lines $3x + 4y - 6 = 0$ and $4x + 7y - 8 = 0$ intersect?

    Answer: Simultaneously solving the two equations we get the intersection as $(2, 0)$.

11. What is the equation of the straight line through the point $(3, 0)$ that is parallel to the straight line $3x - 4y - 6 = 0$?

    Answer: Any line parallel to $3x - 4y - 6 = 0$ may be written as $3x - 4y - c = 0$, where c can be an arbitrary number. If such a line passes through $(3, 0)$ then this point's coordinates must satisfy $3x - 4y - c = 0$.

    In other words, $3 \times 3 - 4 \times 0 - c = 0 \implies c = 9$.

    Therefore, the required equation is $3x - 4y - 9 = 0$.

12. What is the equation of the straight line through the point $(3, 0)$ that is perpendicular to the straight line $3x - 4y - 6 = 0$?

    Answer: Rewrite the given straight line's equation in slope-intercept form: $y = \frac{3}{4}x - \frac{3}{2}$. Its gradient, therefore, is $\frac{3}{4}$. Now the gradient of a straight line that is perpendicular to one of gradient m is $-\frac{1}{m}$.

    Therefore, the gradient of the straight line perpendicular to the given one is $-\frac{4}{3}$ and its equation is of the form $y = -\frac{4}{3}x + c$. Since it passes through $(3, 0)$ we have $0 = -\frac{4}{3} \times 3 + c \implies c = 4$.

Therefore, the required equation is $y = -\frac{4}{3}x + 4$ or $3y + 4x - 12 = 0$.

13. What are the coordinates of the point that is the reflection across the line $y = x$ of the point $(3, 1)$?

    Answer: Reflecting a point across the line $y = x$ interchanges its x and y coordinates. So the reflection of $(3, 1)$ is $(1, 3)$.

14. What is the length of the straight line segment on the plane joining the origin $(0, 0)$ to the point $(3, 4)$? In 3-space (xyz-space) what is the length of the straight line segment joining the points $(1, 2, 3)$ and $(4, 6, 8)$?

    Answer: The (Euclidean) distance between two points (x_1, y_1) and (x_2, y_2) on the plane is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, while that between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in 3-space is (similarly) $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

    Therefore, the length of the straight line segment joining $(0, 0)$ and $(3, 4)$ is $\sqrt{(3 - 0)^2 + (4 - 0)^2} = \sqrt{25} = 5$ and that joining $(1, 2, 3)$ and $(4, 6, 8)$ is

    $$\begin{aligned} & \sqrt{(4-1)^2 + (6-2)^2 + (8-3)^2} \\ = \; & \sqrt{50} = \sqrt{25 \times 2} = \sqrt{25} \times \sqrt{2} = 5 \times 1.414 = 7.07 \end{aligned}$$

    using only the value of $\sqrt{2}$ given above.

15. Determine the value of $\sin 75°$ using only the trigonometric values given at the top of the test (in other words, don't use your calculator to do anything other than arithmetic operations).

    Answer: Use the formula

    $$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

    to write

    $$\begin{aligned} \sin 75° &= \sin(45° + 30°) \\ &= \sin 45° \cos 30° + \cos 45° \sin 30° \\ &= 0.707 \times 0.866 + 0.707 \times 0.5 \\ &= 0.966 \end{aligned}$$

16. What is the dot product (or, scalar product, same thing) of the two vectors u and v in 3-space, where u starts at the origin and ends at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ and v starts at the origin and ends at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{3})$, i.e., $u = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$ and $v = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \sqrt{3}\mathbf{k}$.

    Use the dot product to calculate the angle between u and v.

Answer:
$u \cdot v = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0) \cdot (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{3}) = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + 0 \times \frac{1}{\sqrt{3}} = 1$
Moreover,
$|u| = \sqrt{(\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2 + 0^2} = 1$ and $|v| = \sqrt{(\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2 + (\sqrt{3})^2} = 2$.

Now, $u{\cdot}v = |u||v|\cos\theta$, where θ is the angle between u and v. Therefore, $\cos\theta = \frac{u \cdot v}{|u||v|} = \frac{1}{1\times 2} = \frac{1}{2}$, which means $\theta = 60°$.

17. Determine a vector that is perpendicular to *both* the vectors u and v of the preceding question.

Answer: The cross-product of two (non-zero and non-collinear) vectors is perpendicular to both of them. Now, $u \times v = (\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}) \times (\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \sqrt{3}\mathbf{k}) =$

$$\begin{vmatrix} \mathbf{i} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \mathbf{j} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \mathbf{k} & 0 & \sqrt{3} \end{vmatrix} = \frac{\sqrt{3}}{\sqrt{2}}\mathbf{i} - \frac{\sqrt{3}}{\sqrt{2}}\mathbf{j}$$

Therefore, the vector that starts at the origin and ends at $(\frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{\sqrt{2}}, 0)$ is perpendicular to both u and v (the answer is not unique).

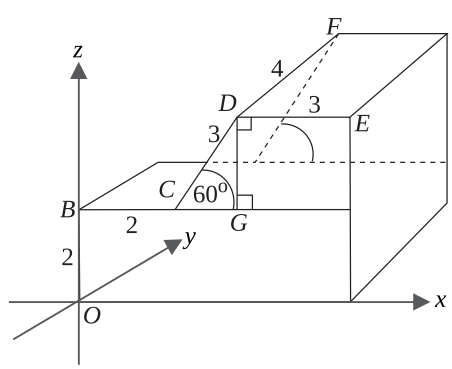

Figure C.2: Solid block (some edges are labeled with their length).

18. For the block in Figure C.2, what are the coordinates of the corner point F?

Answer: Drop the perpendicular DG from D to the straight line through B and C (Figure C.2).

The x-coordinate of F is $|BC|+|CG| = 2+3\cos 60° = 2+3\times 0.5 = 3.5$.

The y-coordinate of F is $|DF| = 4$.

The z-coordinate of F is $|AB|+|GD| = 2+3\sin 60° = 2+3\times 0.866 = 4.598$.

Therefore, $F = (3.5, 4, 4.598)$.

19. For the block again, what is the angle CDE?

    Answer: $\angle CDE = \angle CDG + \angle GDE = 30° + 90° = 120°$.

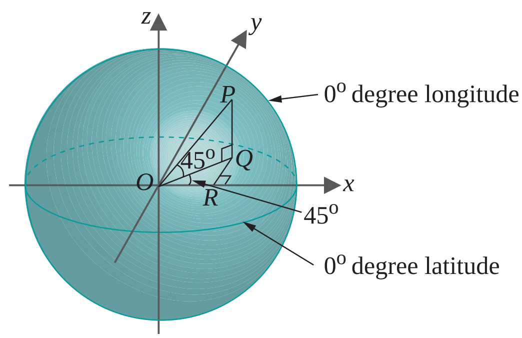

Figure C.3: Unit sphere.

20. For the unit sphere (i.e., of radius 1) centered at the origin, depicted in Figure C.3, the equator (0° latitude) is the great circle cut by the xy-plane, while 0° longitude is that half of the great circle cut by the xz-plane where x-values are non-negative.

    What are the xyz coordinates of the point P whose latitude and longitude are both 45°?

    Answer: Drop the perpendicular from P to Q on the xy-plane and then the perpendicular from Q to R on the x-axis (Figure C.3).

    Now, $|OP| = 1$, so that $|PQ| = 1 \times \sin 45° = \frac{1}{\sqrt{2}}$ and $|OQ| = 1 \times \cos 45° = \frac{1}{\sqrt{2}}$.

    Moreover, $QR = |OQ|\sin 45° = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{1}{2}$ and $OR = |OQ|\cos 45° = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{1}{2}$.

    Now, the x, y and z coordinates of P are $|OR|$, $|QR|$ and $|PQ|$, respectively. Therefore, $P = (\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}})$.

21. Multiply two matrices:

$$\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \end{bmatrix}$$

Answer:

$$\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 10 & 6 & -4 \\ 5 & 4 & -1 \end{bmatrix}.$$

22. Calculate the value of the following two determinants:

$$\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} -1 & 2 & -3 \\ 0 & 5 & -2 \\ 0 & 3 & 3 \end{vmatrix}$$

Answer:

$$\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix} = 2 \times 1 - 4 \times 3 = -10$$

$$\begin{aligned} \begin{vmatrix} -1 & 2 & -3 \\ 0 & 5 & -2 \\ 0 & 3 & 3 \end{vmatrix} &= -1 \times \begin{vmatrix} 5 & -2 \\ 3 & 3 \end{vmatrix} - 0 \times \begin{vmatrix} 2 & -3 \\ 3 & 3 \end{vmatrix} + 0 \times \begin{vmatrix} 2 & -3 \\ 5 & -2 \end{vmatrix} \\ &= -(5 \times 3 - (-2) \times 3) = -21 \end{aligned}$$

23. Calculate the inverse of the following matrix:

$$\begin{bmatrix} 4 & 7 \\ 2 & 4 \end{bmatrix}$$

Answer: To obtain the inverse we have to replace each element by its cofactor, take the transpose and, finally, divide by the determinant of the original matrix.

Replacing each element by its cofactor we get the matrix

$$\begin{bmatrix} 4 & -2 \\ -7 & 4 \end{bmatrix}$$

Taking the transpose next gives

$$\begin{bmatrix} 4 & -7 \\ -2 & 4 \end{bmatrix}$$

Finally, dividing by the determinant $4 \times 4 - 2 \times 7 = 2$ of the original matrix, we have its inverse

$$\begin{bmatrix} 2 & -3.5 \\ -1 & 2 \end{bmatrix}$$

24. If the Dow Jones Industrial Average were a straight-line (or, linear, same thing) function of time (it isn't) and if its value on January 1, 2007 is 12,000 and on January 1, 2009 it's 13,500, what is the value on January 1, 2010?

    Answer: As a linear function of time then the DJIA grows 1500 points in two years, or 750 per year, which takes it to 14,250 on January 1, 2010.

25. Are the following vectors linearly independent?

$$[2\ 3\ 0]^T \qquad [3\ 7\ -1]^T \qquad [1\ -6\ 3]^T$$

    Answer: The vectors are linearly independent if the only solution to the equation

$$c_1[2\ 3\ 0]^T + c_2[3\ 7\ -1]^T + c_3[1\ -6\ 3]^T = [0\ 0\ 0]^T$$

    is $c_1 = c_2 = c_3 = 0$.

    The vector equation above is equivalent to the following set of three simultaneous equations – one from each position in the vectors – in c_1, c_2 and c_3:

$$\begin{aligned} 2c_1 + 3c_2 + c_3 &= 0 \\ 3c_1 + 7c_2 - 6c_3 &= 0 \\ -c_2 + 3c_3 &= 0 \end{aligned}$$

    Solving we find solutions not all 0, e.g., $c_1 = 5$, $c_2 = -3$ and $c_3 = -1$, proving that the given set of vectors is not linearly independent.

26. Determine the linear transformation of $\mathbb{R}^3$ that maps the standard basis vectors

$$[1\ 0\ 0]^T \qquad [0\ 1\ 0]^T \qquad [0\ 0\ 1]^T$$

    to the respective vectors

$$[-1\ -1\ 1]^T \qquad [-2\ 3\ 2]^T \qquad [-3\ 1\ -2]^T$$

    Answer: The required linear transformation is defined by the matrix whose columns are, respectively, the images of the successive basis vectors. In particular, then, its matrix is

$$\begin{bmatrix} -1 & -2 & -3 \\ -1 & 3 & 1 \\ 1 & 2 & -2 \end{bmatrix}$$

27. What is the equation of the normal to the parabola

$$y = 2x^2 + 3$$

at the point (2, 11)?

Answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = 4x$, so at the point (2, 11) the gradient of the tangent is $4 \times 2 = 8$. The gradient of the normal, therefore, is $-\frac{1}{8}$. Its equation, accordingly, is

$$\frac{y - 11}{x - 2} = -\frac{1}{8} \quad \text{or} \quad x + 8y - 90 = 0$$

28. If x and y are related by the equation

$$xy + x + y = 1$$

find a formula for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Answer: Differentiating the equation throughout with respect to x:

$$(\frac{\mathrm{d}}{\mathrm{d}x}x)y + x(\frac{\mathrm{d}}{\mathrm{d}x}y) + \frac{\mathrm{d}}{\mathrm{d}x}x + \frac{\mathrm{d}}{\mathrm{d}x}y = \frac{\mathrm{d}}{\mathrm{d}x}1$$

which simplifies to

$$y + x\frac{\mathrm{d}y}{\mathrm{d}x} + 1 + \frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y + 1}{x + 1}$$

29. The formula for the height at time t of a projectile shot vertically upward from the ground with initial velocity u is

$$h = ut - \frac{1}{2}gt^2$$

assuming only the action of gravitational acceleration g (ignoring wind resistance and other factors).

What is the velocity of the projectile at time t? What is the maximum height reached by the projectile?

Answer: Its velocity at time t is

$$\frac{\mathrm{d}h}{\mathrm{d}t} = u - gt$$

When the projectile attains maximum height, its velocity is 0. Therefore, $u - gt = 0$, implying that $t = \frac{u}{g}$. Therefore, the maximum height reached by the projectile is obtained by substituting $t = \frac{u}{g}$ in the formula for its height, which gives the maximum height as

$$u\frac{u}{g} - \frac{1}{2}g\left(\frac{u}{g}\right)^2 = \frac{u^2}{g} - \frac{u^2}{2g} = \frac{u^2}{2g}$$

30. At what points do the curves $y = \sin x$ and $y = \cos x$ meet for values of x between 0 and 2π? What angles do they make at these points?

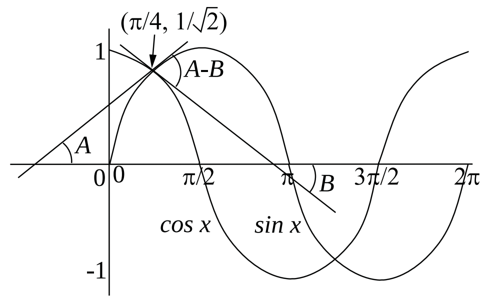

Figure C.4: Graphs of $\sin x$ and $\cos x$ (*sketch, not exact*).

Answer: When the curves (see Figure C.4) meet, their y-values are equal, so $\sin x = \cos x$, giving $\tan x = \frac{\sin x}{\cos x} = 1$. The two values in the interval $[0, 2\pi]$ where $\tan x = 1$ are $\frac{\pi}{4}$ and $\frac{5\pi}{4}$. Therefore, the two points at which the curves meet are $(\frac{\pi}{4}, \frac{1}{\sqrt{2}})$ and $(\frac{5\pi}{4}, -\frac{1}{\sqrt{2}})$ $(\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\cos\frac{5\pi}{4} = \sin\frac{5\pi}{4} = -\frac{1}{\sqrt{2}})$.

For the first curve $\frac{dy}{dx} = \cos x$, while for the second $\frac{dy}{dx} = -\sin x$. At the point of intersection $(\frac{\pi}{4}, \frac{1}{\sqrt{2}})$, therefore, the gradients of the two curves are $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $-\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$, respectively.

If the tangent lines at $(\frac{\pi}{4}, \frac{1}{\sqrt{2}})$ make angles A and B, respectively, with the x-axis, then the gradients are precisely the tan of these angles. Therefore, $\tan A = \frac{1}{\sqrt{2}}$ and $\tan B = -\frac{1}{\sqrt{2}}$ (B is a negative angle). The angle between the curves – which by definition is the angle between their tangents – at $(\frac{\pi}{4}, \frac{1}{\sqrt{2}})$ is $A - B$. Now,

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}} = 2\sqrt{2}$$

which means $A - B = \tan^{-1} 2\sqrt{2} = \tan^{-1} 2.828 = 1.231$ radians or $70.526°$ (approximately). The angle between the curves at the other point of intersection is the same by symmetry.